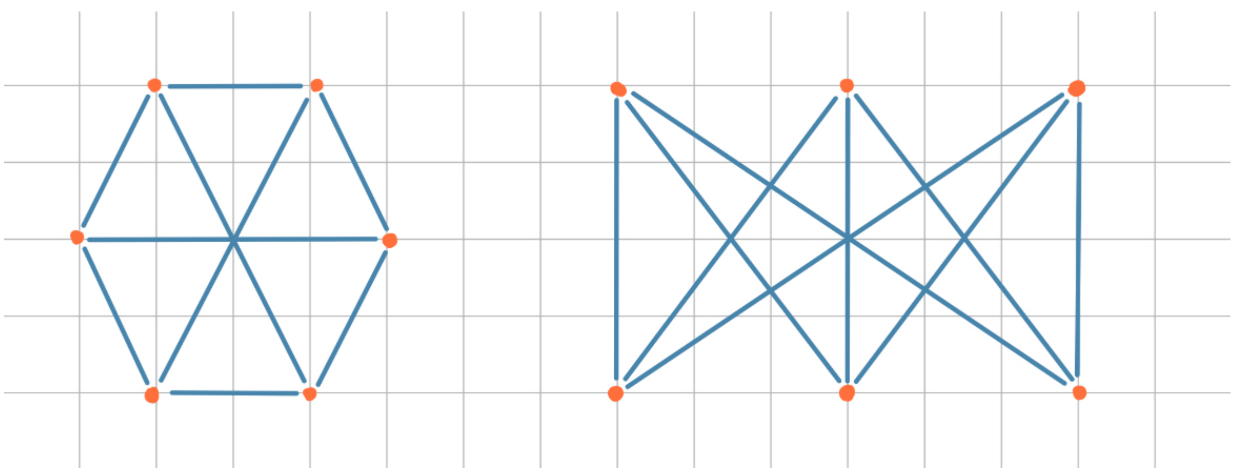
Grade 7/8 Math Circles
February 19th - 23rd, 2024
Graph Theory: Isomorphisms

Section 1: Graph Theory

Graph theory is a branch of mathematics which studies collections of points (called vertices) which are connected to each other with lines (called edges), in this way we can think of vertices as objects and edges as the relationships between certain objects. Graphs like the ones below can be used to visualize traffic and computer networks, and can be use to find optimal solutions to a variety of problems.

Beyond its real world applications, graph theory is a great way to begin to understand what a function is an how its properties play into different areas of mathematics. Today we'll look at relationships between graphs through the lens of functions! At first glance the two graphs below do not look at all related but after staring for a while we can see how we could be able to reshape one into the other.



Stop and Think

Why do you think relationships between graphs could be of benefit to the people who are working with them?

Figuring out and proving that two graphs are in reality the same graph taking on two different forms is not a simple task, but through this lesson we will learn about functions and how two things that may appear completely unrelated can actually have an underlying relationship which when uncovered can simplify the answers to other questions we may have.

Section 2: Graphs & Their Properties

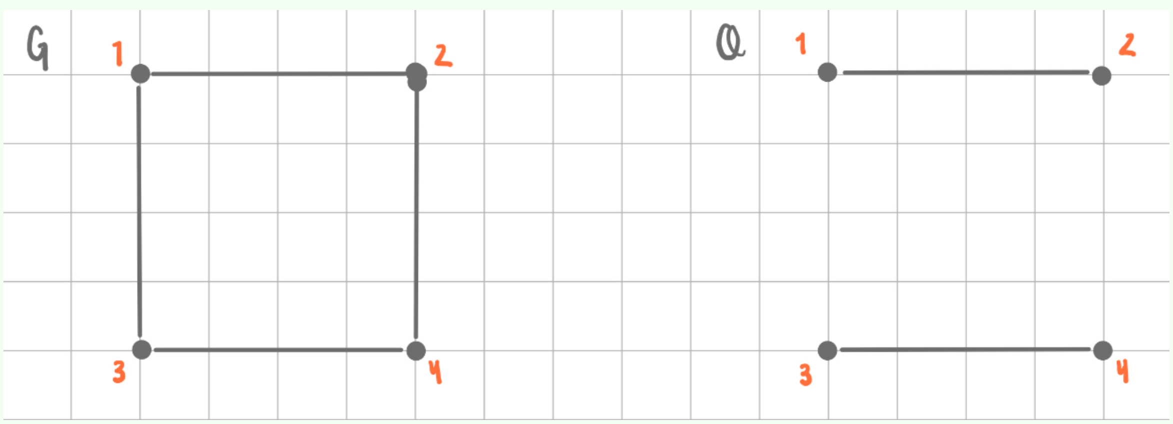
Definition 1

A **Graph** G is a set of points called vertices together with a set of lines called edges which connect specific pairs of vertices. We write $V(G)$ to express the set of vertices and $E(G)$ to express the set of edges.

Let's look at how we express G, $V(G)$, and $E(G)$ with an example.

Example 2.1

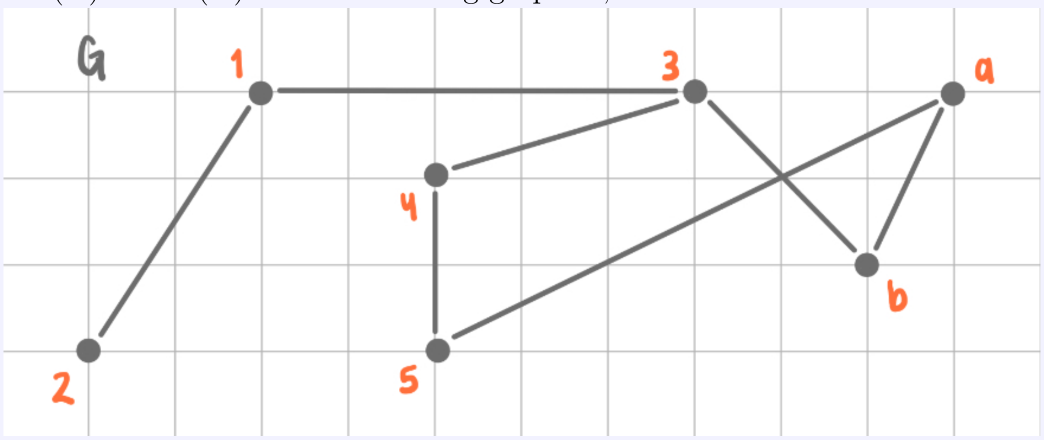
What are $V(G)$, $E(G)$, $V(Q)$, and $E(Q)$ for the graphs G and Q below?



Solution: $V(G)$ is the set of vertices of the graph G (graph on the left hand side), we denote $V(G)$ as $V(G) = \{1, 2, 3, 4\}$. $E(G)$ is the set of edges of G and we denote $E(G)$ as $E(G) = \{\{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}\}$. Similarly we have; $V(Q) = \{1, 2, 3, 4\}$ and $E(Q) = \{\{1,2\}, \{3,4\}\}$

Exercise 2.1

Determine $V(G)$ and $E(G)$ for the following graph G;



Terminology

Besides the notation $V(G)$ and $E(G)$ that we have already learned, there are many more terms that help us speak about graphs in a more clear way, let's look at some of this terminology!

Definition 2

We say two vertices are **adjacent** if there is an edge connecting them. For example in Exercise 2.1 the vertices 3 and b are adjacent, but 5 and b are not.

Definition 3

For a given vertex v in a graph G the **neighbours** of v are all vertices adjacent to v. For example in Exercise 2.1 the neighbours of the vertex 3 are; $\{1, 4, b\}$.

Definition 4

For a given vertex v in a graph G the **degree** of v is the number of neighbours v has. For example in Exercise 2.1 the degree of the vertex 3 is 3.

Exercise 2.2

Build a graph G with more than 5 vertices and more than 6 edges. Write out $V(G)$, $E(G)$, the neighbours of each vertex, and the degrees of each vertex.

Section 3: Functions and Isomorphisms

Now that we've established some basic terminology that will help us speak about graphs, we shift gears a bit to look at the following two graphs. What do you notice about them?

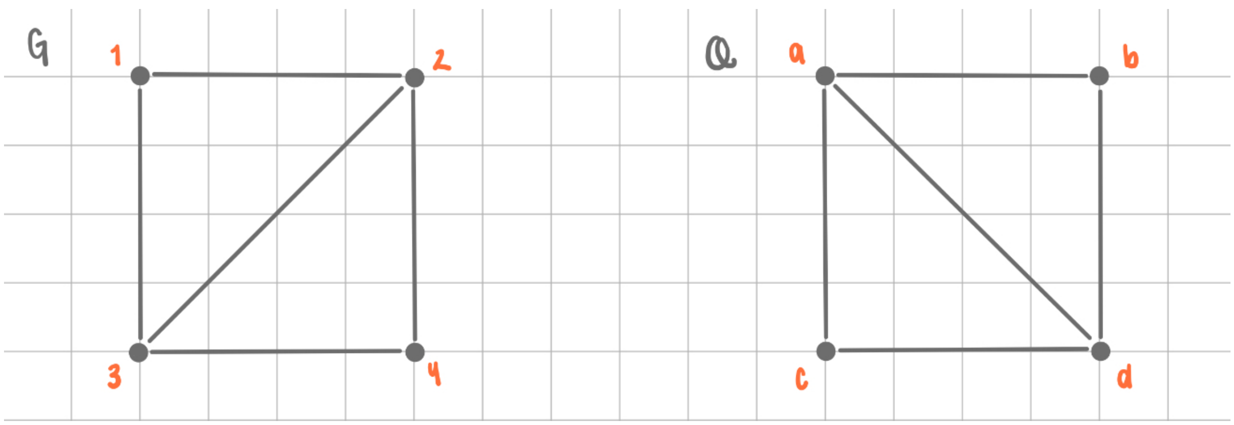


Figure 1: Toy Graphs

After carefully observing these two graphs we can see that they are mirror images of one another which simply have their vertices labelled in different ways. For now we'll say that these graphs are related, but how could we prove that G and Q are in fact the "same" graph? This is where we will introduce the idea of a Graph Relabelling.

Definition 5

We will say that two graphs are **related** if we can find a way to relabel the vertices of one graph to match the vertices of the second such that the degrees and adjacencies of both graphs are preserved. Such a relabelling is called a **valid graph relabelling**.

If we cannot find a way to relabel our vertices such that the above holds then we will say that the graphs are not related.

Example 3.1

For the graphs in Figure 1 why is the relabelling $1 \to a$ (this is read as 1 is mapped to a), $2 \to d$, $3 \to b$, $4 \to c$ invalid?

Solution:

From Definition 5 we know that a relabelling is invalid if the degrees and adjacencies of both graphs are **not** preserved. So let's see where this relabelling is failing by building a table to compare degrees of the relabelled vertices:

Relabelling	Degree of 1st Vertex	Degree of 2nd Vertex
$1 \to a$	2	3
$2 \to d$	3	3
$3 \to b$	3	2
$4 \to c$	2	3

From the above we can see that the degrees of the vertices 1 and a, as well as the degrees of the vertices 4 and c are not equal, this gives us the first reason for which the relabelling is invalid. Let's now see if adjacencies are preserved;

Relabelling	Neighbours of 1st Vertex	Relabelled Neighbours	Neighbours of 2nd Vertex
$1 \to a$	2 and 3	d and b	b, c and d
$2 \to d$	1, 3, and 4	a, b and c	a, b and c
$3 \to b$	1, 2 and 4	a, d and c	a and d
$4 \to c$	2 and 3	d and b	a and d

Once again from the above we can see that the only relabelling that preserves adjacencies correctly is $2 \to d$, but all other relabellings are incorrect, this gives us the second reason for which the relabelling is invalid. Therefore we've shown that this relabelling does not preserve the degrees and adjacencies of both graphs and thus it is invalid.

Exercise 3.1

Give another example of an invalid relabelling with respect to the graphs in Figure 1, also give an example of a valid relabelling.

Functions

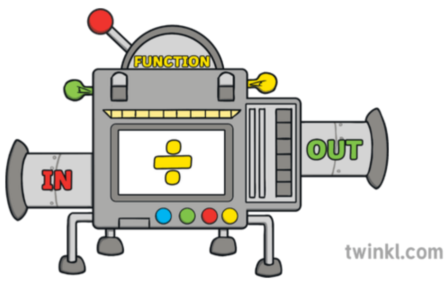


Figure 2: Function Input-Output

Now to have a more formal way to speak about what we've been calling "relabellings" up until this point we will introduce the formal definition of a function! Functions will help us in introducing properties which will make the process of determining if a relabelling/function is valid much more straight forward than the process we went through in Example 3.1, and they will introduce us to some fundamental properties of functions! Imagine f working like a machine (as seen in Figure 2), where f is "fed" some input and it will modify it in some uniform way to give us a corresponding output.

Definition 6

A **function** f is defined as a relation between a set of inputs and a set of outputs, which relates each input to exactly one output. Some useful notation is as follows:

- If S is our set of inputs and T is our set of outputs then we say "f maps from S to T" and write $f : S \to T$.
- Let s be an input from S and let t be its corresponding output, then we say "f of s is equal to t" and write $f(s) = t$.

Example 3.2

Build an input-output table for the following function g: $\{1, 2, 3, 4\} \to \{48, 3, 27, 12\}$ where $g(s) = 3 \times s^2$ where s is an element in $\{1, 2, 3, 4\}$.

Solution: An input-output table is a t-chart which will contain the elements $\{1, 2, 3, 4\}$ in its left column and in the right column we will fill in their corresponding value $g(s)$, as follows:

s	$g(s)$
1	3
2	12
3	27
4	48

Example 3.3

Which of the following are functions, and if it is a function what is the operation?

s	$g(s)$
1	2
2	4
3	6
4	8

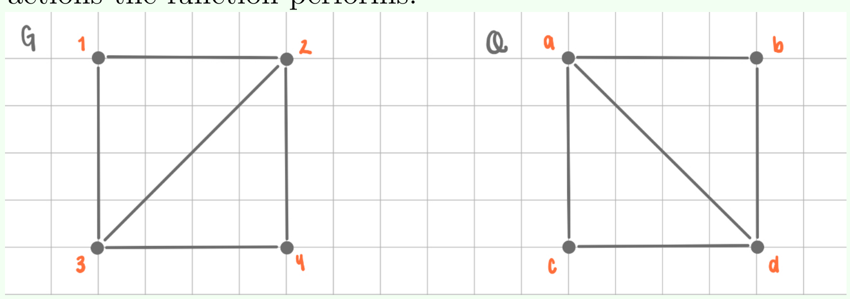
x	$f(x)$
1	1
2	2
1	3
4	4

m	$h(m)$
-2	2
-1	1
1	1
2	2

Solution: Table 1 and Table 3 are the tables which describe functions, g acts on its inputs by doubling their value and h acts on its inputs by multiplying them by -1. f is not a function because the input 1 corresponds to both the outputs 1 and 3.

Example 3.4

Consider the relabelling of G to Q; $1 \to a$, $2 \to d$, $3 \to b$, $4 \to c$. Identify the set of inputs, outputs and the actions the function performs.



Solution: The set of inputs is the set of the elements we are relabelling, this is $V(G)$ and set of outputs is $V(Q)$, so we can write $f : V(G) \to V(Q)$. Now to write down the actions the function performs on its inputs we can use an input-output table as shown:

v	1	2	3	4
$f(v)$	a	d	b	c

Valid Relabellings & Isomorphisms

Earlier we looked at the idea of relabellings being valid, and we defined a valid relabelling as a relabelling in which the degree and adjacencies of a vertex are preserved once relabelled. Now that we have function notation it is easier to create criteria for all valid relabellings, which we will formally call graph isomorphisms!

Exercise 3.3

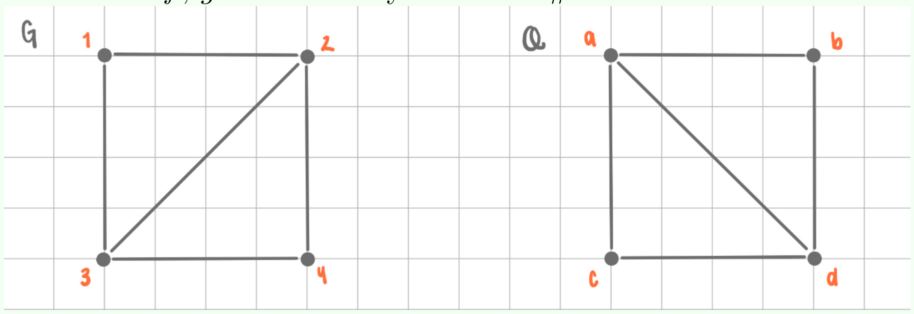
Brainstorm what are some criterion which a function needs to fulfill in order to represent a valid relabellings (i.e when is a function a graph isomorphism?).

Throughout the lesson we've been implicitly building up a criteria for when a relabelling is valid and when it is not, and we based its validity on the preservation of certain properties of the vertices. The first criterion we've been using the most throughout our journey with graph isomorphisms so far is that fact that we want to make sure that the adjacencies and degree of a given vertex are preserved after being relabelled. This brings us to our first criterion for graph isomorphisms.

Criterion #1 of Graph Isomorphisms

Let G and Q be graphs and let $f : V(G) \to V(Q)$ be a function. Suppose that u and v are adjacent vertices in G, then in order to preserve this adjacency we require that $f(u)$ and $f(v)$ are adjacent in Q.

Example 3.5
Let $f : V(G) \to V(Q)$, $g : V(Q) \to V(G)$ and $h : V(G) \to V(Q)$ be functions whose actions are given below, determine if f, g and h satisfy Criterion #1.



v	1	2	3	4
$f(v)$	a	b	c	d

u	a	b	c	d
$g(u)$	2	1	4	3

v	1	2	3	4
$h(v)$	b	a	b	d

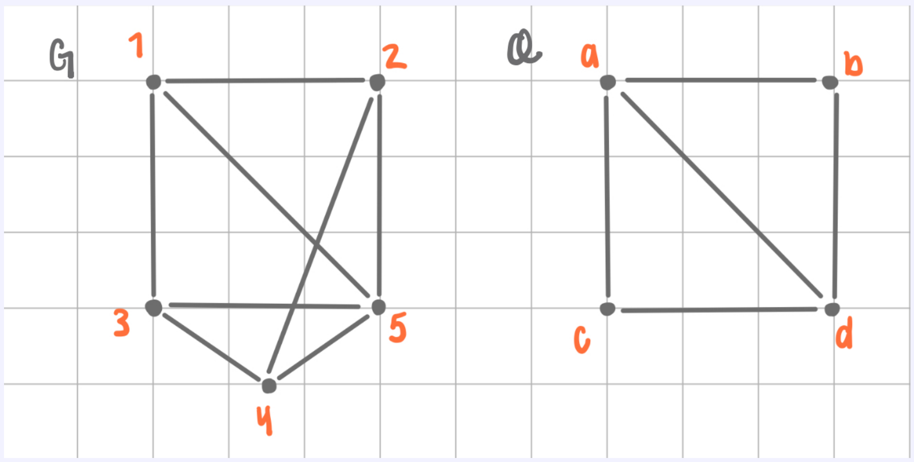
Solution: Let's look at each function one by one;

1. We need to check that if u and v are adjacent in G then $f(u)$ and $f(v)$ are adjacent in Q. We can see that this immediately fails because 3 is adjacent to 2 in G but $f(3) = c$ is not adjacent to $f(2) = b$ in Q.
2. We need check that if u and v are adjacent in Q then $g(u)$ and $g(v)$ are adjacent in G. After some inspection we see that this holds, thus criteria 1 is satisfied.
3. We need to check that if U and V are adjacent in G then $h(u)$ and $h(v)$ are adjacent in Q. We can see that this immediately fails because 1 is adjacent to 3 in G but $h(1) = b$ is not adjacent to $h(3) = b$ in Q (since no vertex is adjacent to itself).

In the last example we saw a function $h : V(G) \to V(Q)$ where two vertices in G were relabelled as the same vertex in Q, lets explore this a bit more;

Exercise 3.4

Consider the graphs G, Q and the function $f : V(G) \to V(Q)$ given below:



v	1	2	3	4	5
$f(v)$	a	b	c	a	d

In the following table we've shown that all adjacencies have been preserved under this relabelling, does this mean that $f : V(G) \to V(Q)$ is an isomorphism? Justify your answer.

$f(v) = u$	Neighbours of v	Relabelled Neighbours	Neighbours of u
$f(1) = a$	2, 3 and 5	b, c and d	b, c and d
$f(2) = b$	1, 4 and 5	a, a and d	a and d
$f(3) = c$	1, 4 and 5	a, a and d	a and d
$f(4) = a$	2, 3 and 5	b, c and d	b, c and d
$f(5) = d$	1, 2, 3, 4	a, b, c and a	a, b and c

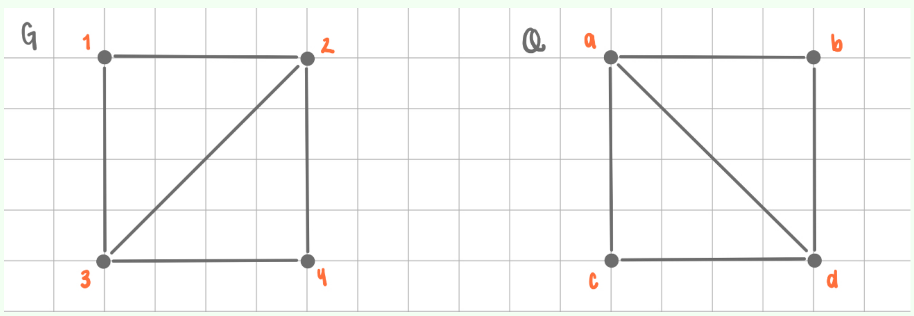
As we saw in Exercise 3.4, $f : V(G) \to V(Q)$ ended up not being as isomorphism because even though the adjacency relationships of each vertex were preserved, the degree of each vertex was not since we had two distinct vertices 1 and 4 both being relabelled as a. This brings us to our second criterion!!

Criterion #2 of Graph Isomorphisms

Let G and Q be graphs, let $f : V(G) \to V(Q)$ be a function and let u and v be vertices in G, if $f(u) = f(v)$ then we require that $u = v$. When a function f satisfies Criterion #2 we say f is injective.

Example 3.6

Let $f : V(G) \to V(Q)$, $g : V(Q) \to V(G)$ and $h : V(G) \to V(Q)$ be functions whose actions are given below, determine if f, g and h satisfy Criterion #2.



v	1	2	3	4
$f(v)$	a	b	c	d

u	a	b	c	d
$g(u)$	2	1	4	3

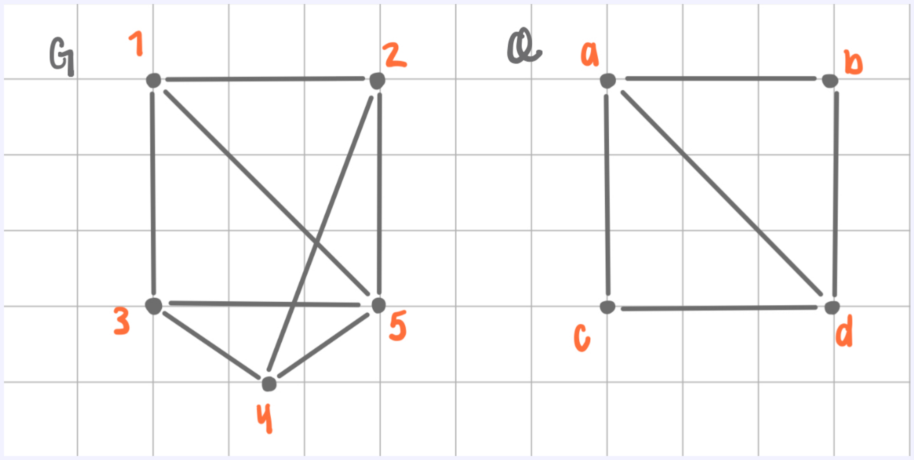
v	1	2	3	4
$h(v)$	b	a	b	d

Solution: Let's look at each function one by one;

1. On the bottom row of f's input-output chart we see that no output (letter) is repeated this means that each vertex in G is being mapped to a unique output, and so f satisfies Criterion #2 equivalently we say f is injective.
2. On the bottom row of g's input-output chart we see that no output (number) is repeated this means that each vertex in Q is being mapped to a unique output, and so g is injective.
3. We can quickly identify that the vertex b appears twice on bottom row of h's input-output, upon further inspection we see that $h(1) = b = h(3)$ but $1 \neq 3$ so h is not injective.

Exercise 3.5

Consider the graphs G, Q and the function $g : V(Q) \to V(G)$ given below:



v	a	b	c	d
$g(v)$	3	2	4	5

We can see that $g : V(Q) \to V(G)$ satisfies Criterion #2 does this mean that $g : V(Q) \to V(G)$ is an isomorphism? Justify your answer.

As we saw in Exercise 3.5, $g : V(Q) \to V(G)$ ended up not being as isomorphism because even though g is injective, there was no vertex in Q which was relabelled as 5, which caused some issues. This brings us to our third and final criterion!!

Criterion #3 of Graph Isomorphisms

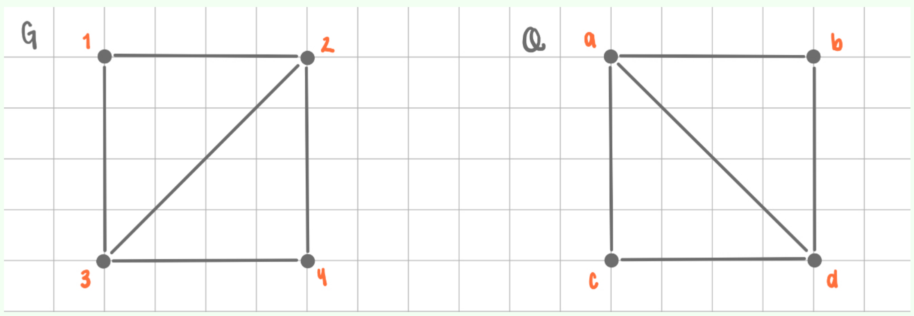
Let G and Q be graphs, let $f : V(G) \to V(Q)$ be a function and let u be a vertex in Q. We require that there exists a vertex v in G such that $f(v) = u$. When a function f satisfies Criterion #3 we say f is surjective.

Criteria of Graph Isomorphisms

Let G and Q be graphs and let $f : V(G) \to V(Q)$ be a function, if f satisfies all 3 criterion then we say that f is an **isomorphism** and that G and Q are **isomorphic**.

Example 3.7

Let $f : V(G) \to V(Q)$, $g : V(Q) \to V(G)$ and $h : V(G) \to V(Q)$ be functions whose actions are given below, determine if f, g and h satisfy Criterion #3.



v	1	2	3	4
$f(v)$	a	b	c	d

u	a	b	c	d
$g(u)$	2	1	4	3

v	1	2	3	4
$h(v)$	b	a	b	d

Solution: Let's look at each function one by one;

1. On the the bottom row of f's input-output chart we can see that every vertex in $V(Q)$ is listed, this means that for every vertex in $V(Q)$ there exists a vertex in $V(G)$ which is being mapped to it, and so f satisfies Criterion #3 equivalently we say f is surjective.

2. On the the bottom row of g's input-output chart we can see that every vertex in $V(G)$ is listed, this means that for every vertex in $V(G)$ there exists a vertex in $V(Q)$ which is being mapped to it, and so g is surjective.

3. On the the bottom row of h's input-output chart we can see that the vertex c in $V(Q)$ is not listed, this means that there does not exist a vertex in $V(G)$ which is being mapped to c, and so h is not surjective.

Through out Examples 3.5 to 3.7 we collected the following results;

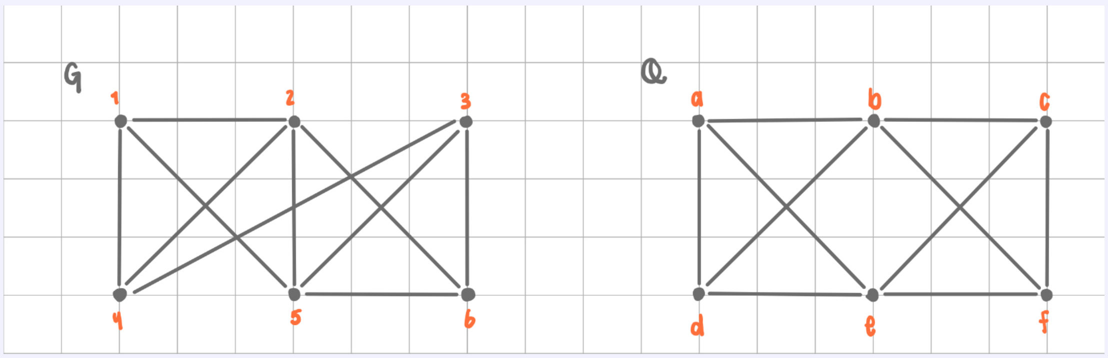
- $f : V(G) \to V(Q)$ satisfies Criterion #2 and #3 but it did not satisfy Criterion #1, therefore we can conclude that f is **not** an isomorphism.

- $g : V(Q) \to V(G)$ satisfied all of Criterion #1, 2 and 3, therefore we can conclude that g is an isomorphism and thus G and Q are isomorphic.

- $h : V(G) \to V(Q)$ did not satisfy any of Criterion #1, 2 and 3 therefore we can conclude that h is **not** an isomorphism.

Exercise 3.6

Does there exist an isomorphism between the graphs G and Q as seen below?



If yes give an isomorphism, if not explain why.